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De Sitter Quasigroups

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The Cayley–Klein parameters for the de Sitter groups $SO(4, 1)$ and $SO(3, 2)$ are introduced, and in an extension of the earlier investigation of quasigroups connected with Clifford groups, quasigroups connected with the $SO(4, 1)$ and $SO(3, 2)$ groups are determined. It is shown that these quasigroups have eight-dimensional, double-valued irreducible cracovian representations. The covariance of a five-dimensional form of the Dirac equation with respect to the quasi-rotations forming quasigroups connected with the groups $SO(4, 1)$ and $SO(3, 2)$ is demonstrated. An analogy is drawn between Weyl's hidden symmetry group and a quasigroup.

1. INTRODUCTION

The investigations of quasigroups in the mathematical literature (Chein *et al.*, 1990; Pflugfelder, 1990; Sabinin, 1999) have a counterpart in physics in the attempts of applying nonassociative algebras in quantum mechanics, which were initiated by Jordan (1932) and Jordan *et al.* (1934) and continued in the papers by Segal (1947) and Sherman (1956) and extended on elementary particle physics by Gürsey (1979). A survey of papers on nonassociative geometry with the reference to space-time was recently presented by Sabinin (2001). The line of thought of that survey was pursued by Sbitneva (2001) in an application of nonassociative geometry to special relativity. Nonassociative gauge theory is the subject of a recent paper by Nesterov (2001).

In an earlier paper on this subject (Kociński, 2001) a certain type of quasigroup connected with Clifford groups was defined. In Clifford groups generated by N elements $\gamma_1, \gamma_2, \ldots, \gamma_N$ for $N = 1, 2, \ldots,$ which fulfil the condition

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \quad \mu, \nu = 1, 2, \ldots, N \tag{1}$$

two group automorphisms were considered: (1) the involution operation I defined by

$$I(\gamma_\mu) = -\gamma_\mu, \quad I(I(\gamma_\mu)) = \gamma_\mu, \quad I(\pm 1) = \pm 1$$

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$$I(\gamma_\mu \gamma_\nu \cdots \gamma_\sigma) = I(\gamma_\sigma) \cdots I(\gamma_\nu) I(\gamma_\mu), \tag{2}$$
$$\gamma_\mu, \gamma_\nu, \ldots, \gamma_\sigma \in G$$

and (2) the automorphism defined by the equality

$$I(\gamma_A)\gamma_B = \gamma_C, \quad \text{for a fixed } \gamma_A \tag{3}$$

where, for brevity, γ_A, γ_B, and γ_C denote arbitrary elements of the Clifford group, i.e. also arbitrary products of the elements $\gamma_\alpha, \alpha = 1, 2, \ldots, N,$ where for any $\gamma_A = \gamma_\alpha \gamma_\beta \cdots \gamma_\varrho$ we have $\gamma_A^{-1} = \gamma_\varrho \cdots \gamma_\beta \gamma_\alpha$. With the notation

$$\gamma_\mu \gamma_\nu \cdots \gamma_\sigma \equiv \gamma_{\mu\nu\cdots\sigma} \tag{4}$$

the following product was defined:

$$(\gamma_{\mu\nu\cdots\sigma}) \cdot (\gamma_{\epsilon\eta\cdots\rho}) := I(\gamma_{\epsilon\eta\cdots\rho})\gamma_{\mu\nu\cdots\sigma} \tag{5}$$

where the product on the right-hand side is the associative product in Clifford groups. With a, b, and c denoting three γ-symbols undergoing the "dot" product in Eq. (5) it was found that

$$(a \cdot b) \cdot c = a \cdot (c \cdot \tau b) \tag{6}$$

where τ denotes the right unit element. The symbol τ replaces the symbol $\hat{1}$ which in Kociński (2001) denotes the right unit element in the defined particular type of quasigroups.

These quasigroups, which were called nonassociative groups, have representations. They are analogous to the matrix representations of groups provided that the "row-by-column" multiplication of two matrices is replaced by the "column-by-column" product. This type of product of matrices was introduced by Banachiewicz (1929, 1937, 1938, 1959), and matrices undergoing the "column-by-column" multiplication were called by him "cracovians." The cracovian algebra was presented by Sierpiński (1951). The nonassociative group of quasi-rotations connected with rotations belonging to the proper orthochronous Lorentz group was defined and a four-dimensional, double-valued irreducible cracovian representation of that quasigroup was determined (Kociński, 2001). That investigation will now be extended to quasigroups connected with two five-dimensional, pseudo-orthogonal rotation groups, i.e. to the de Sitter groups $SO(4, 1)$ and $SO(3, 2)$. These groups were investigated by a number of authors. The references to papers concerning the pseudo-orthogonal rotation groups and their contractions may be found in Philips and Wigner (1968). These authors discuss various properties of the de Sitter groups, the physical interpretation of the group $O(4, 1)$, and, in particular, the question how the positive nature of energy can be incorporated into that interpretation. The relation of the $SO(4, 1)$ group with the respective Clifford group algebra was investigated by Gürsey (1964). He determined the irreducible representation (irrep) of the group $SO(4, 1)$ depending on the 10 rotation angles. The question of the

Cayley–Klein parameters of the de Sitter groups $SO(4, 1)$ and $SO(3, 2)$ was not considered in the literature of the subject.

We will extend the investigation of Gürsey (1964) by determining in Section 2 the Cayley–Klein parameters for the $SO(4, 1)$ and $SO(3, 2)$ groups. In this we will follow the method of Sommerfeld (1944), who derived the Cayley–Klein parameters for rotations in the Minkowski space. The motivation behind this part of the paper is that the earlier applied method (Kociński, 2001) of determination of a double-valued cracovian irrep of quasigroup depends on the knowledge of the Cayley–Klein parameters of the proper orthochronous Lorentz rotations. The extension of that mode of reasoning to the quasigroups connected with the de Sitter groups therefore hinges on the knowledge of the respective Cayley–Klein parameters.

In Section 3 we determine the quasigroups of quasi-rotations connected with the rotation groups $SO(4, 1)$ and $SO(3, 2)$, and in Section 4 we calculate the eight-dimensional, double-valued cracovian irreps of these quasigroups. These are analogous to the four-dimensional, double-valued cracovian irrep of the quasi-group of quasi-rotations connected with the proper orthochronous Lorentz group (Kociński, 2001).

In Section 5 the covariance of a five-dimensional form of the Dirac equation under the quasi-rotations belonging to the quasigroups connected with the $SO(4, 1)$ and $SO(3, 2)$ groups, respectively, is demonstrated. This also means the covariance of the Dirac equation under the quasi-rotations belonging to the quasigroup connected with the proper orthochronous Lorentz group.

In Section 6 a tentative analogy is drawn between Weyl's hidden symmetry group of an object (Weyl, 1952) and a quasigroup connected with a group. The symmetry group of the Dirac equation is discussed in this respect.

2. CAYLEY–KLEIN PARAMETERS OF FIVE-DIMENSIONAL ROTATIONS

In the description of rotations in pseudo-orthogonal spaces of metric signatures (4, 1) or (3, 2), i.e. of four real and one imaginary or three real and two imaginary dimensions, we will utilize the Clifford group algebra, generated by the elements γ_1, γ_2, γ_3, and γ_4 which obey the defining condition in Eq. (1) with $\mu, \nu = 1, 2, 3, 4$. Introducing the fifth element $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, which also fulfils Eq. (1), we can say that the five elements γ_1, γ_2, γ_3, γ_4, and γ_5 generate the Clifford group algebra relative to the pseudo-Euclidean spaces $E(4, 1)$ or $E(3, 2)$. To the elements $\gamma_1, \ldots, \gamma_5$ the orthogonal axes $x_1, \ldots, x_5$, respectively, can be attached. We follow Sommerfeld (1944) and write a vector in five-dimensional, pseudo-orthogonal space in the form

$$\vec{x} = x_\mu \gamma_\mu \tag{7}$$

where the coordinates x_1, x_2, x_3 are real, $x_4 = ict$, with c denoting the speed of light in the vacuum and t denoting time, and the x_5 coordinate is real or imaginary in $E(4, 1)$ or $E(3, 2)$ space, respectively. For the products of two γ-elements we introduce the notation

$$\gamma_\mu \gamma_\nu \stackrel{\text{def}}{=} \gamma_{\mu\nu} = -\gamma_{\nu\mu} \quad \mu, \nu = 1, \ldots, 5 \tag{8}$$

Rotations of a four-vector in Minkowski space, with $x_4 = ict$, were expressed by Sommerfeld (1944) by the Cayley–Klein parameters in the form

$$\vec{x}' = x'_\alpha \gamma_\alpha = S_8^{-1}(x_\alpha \gamma_\alpha) S_8 \tag{9}$$

with the biquaternion

$$S_8 = A\gamma_{23} + B\gamma_{31} + C\gamma_{12} + D + a\gamma_{14} + b\gamma_{24} + c\gamma_{34} - d\gamma_5 \tag{10}$$

and the inverse biquaternion

$$S_8^{-1} = -A\gamma_{23} - B\gamma_{31} - C\gamma_{12} + D - a\gamma_{14} - b\gamma_{24} - c\gamma_{34} - d\gamma_5 \tag{11}$$

where the Cayley–Klein parameters A, B, C, and D are real and a, b, c, and d are imaginary. We have $S_8\, S_8^{-1} = S_8^{-1}\, S_8 = 1$ on the two conditions

$$A^2 + B^2 + C^2 + D^2 + a^2 + b^2 + c^2 + d^2 = 1 \tag{12}$$

$$Aa + Bb + Cc + Dd = 0 \tag{13}$$

Equations (9)–(11) for rotations in the Minkowski space can be extended on five-dimensional rotations in $E_{4,1}$ and $E_{3,2}$ spaces. We will demonstrate that they are determined by the formula

$$\vec{x}' = x'_\mu \gamma_\mu = S_{16}\,(x_\mu\, \gamma_\mu) S_{16}^{-1} \tag{14}$$

where

$$S_{16} = S_8 + u\gamma_{15} + v\gamma_{25} + w\gamma_{35} + Z\gamma_{45} - U\gamma_1 - V\gamma_2 - W\gamma_3 - z\gamma_4 \tag{15}$$

and

$$S_{16}^{-1} = S_8^{-1} - u\gamma_{15} - v\gamma_{25} - w\gamma_{35} - Z\gamma_{45} - U\gamma_1 - V\gamma_2 - W\gamma_3 - z\gamma_4 \tag{16}$$

with the transformations S_8 and S_8^{-1} given in Eqs. (10) and (11), where U, V, W, Z and u, v, w, z are parameters. From the condition

$$S_{16}^{-1} S_{16} = 1 \tag{17}$$

there follow six conditions for the 16 parameters appearing in Eqs. (15) and (16):

$$\begin{aligned} A^2 + B^2 + C^2 + D^2 + a^2 + b^2 + c^2 + d^2 + U^2 \\ + V^2 + W^2 + Z^2 + u^2 + v^2 + w^2 + z^2 = 1 \end{aligned} \tag{18}$$

$$Aa + Bb + Cc + Dd + Uu + Vv + Ww + Zz = 0 \tag{19}$$

$$AZ + BW - CV + DU - az - bw + cv - du = 0 \tag{20}$$

$$AW - BZ - CU - DV - aw + bz + cu + dv = 0 \tag{21}$$

$$AV - BU + CZ + DW - av + bu - cz - dw = 0 \tag{22}$$

$$Au + Bv + Cw - Dz - aU - bV - cW + dZ = 0 \tag{23}$$

Equations (19)–(23) represent a system of five linear inhomogeneous equations for the five parameters d, U, V, W, and z in terms of the parameters A, B, C, a, b, c, u, v, w, Z, and D. The determinant Δ of this system of equations is equal to

$$\begin{aligned}\Delta = D[&D^2(D^2 + A^2 + B^2 + C^2 + a^2 + b^2 + c^2 + u^2 + v^2 + w^2 + Z^2)\\ &+ (Aa + Bb + Cc)^2 + (AZ - bw + cv)^2\\ &+ (CZ - av + bu)^2 + (Au + Bv + Cw)^2]\end{aligned} \tag{24}$$

It can be shown that this determinant cannot vanish if Eqs. (19)–(23) are fulfilled. The parameters d, U, V, W, and z therefore are uniquely determined by these five equations. The parameter D then is determined from Eq. (18).

We notice that the inverse transformation S_{16}^{-1} in Eq. (16) is obtained from S_{16} in Eq. (15) by reversing the signs of the 10 rotational parameters: $A, B, C, a, b, c, u, v, w, Z$. Consequently, if instead of Eq. (17) the condition $S_{16}S_{16}^{-1} = 1$ were used, we again would have obtained the six conditions in Eqs. (18)–(23), in which the above 10 parameters would have appeared with reversed signs. The determinants of the matrices connected with the transformations S_{16} and S_{16}^{-1} are equal to 1 for both sets of the six conditions. By analogy with rotations in Minkowski space we assume that for $E_{4,1}$ space, which is Minkowski space extended by one real dimension, we have u, v, w real and Z imaginary. We now turn to Eq. (19). Its first four terms are imaginary, since A, B, C, D are real and a, b, c, d are imaginary. Consequently, the remaining four terms also must be imaginary, since otherwise Eq. (19) would split into two conditions and this is unacceptable as the total number of conditions cannot exceed six. Therefore if u, v, w are real and Z is imaginary, then U, V, W have to be imaginary and z has to be real, and vice versa. We thus conclude that in the case of $E_{4,1}$ space we have

$$\begin{array}{ll} u, v, w, z & \text{real} \\ U, V, W, Z & \text{imaginary} \end{array} \tag{25}$$

and for $E_{3,2}$ space

$$\begin{array}{ll} u, v, w, z & \text{imaginary} \\ U, V, W, Z & \text{real} \end{array} \tag{26}$$

It can be verified that the product of two transformations of the type S_{16} yields another transformation of that type, i.e. with the resultant parameters fulfilling Eqs. (18)–(23). The identity transformation is obtained with $D = 1$ and the remaining 15 parameters equal to zero. The transformations S_{16} therefore constitute a continuous group depending on 10 independent parameters. The irreducible four-dimensional matrix form of the transformation S_{16} was given by Kociński (2000).

To show that Eq. (14) determines a rotation it suffices to verify that $(\vec{x}')^2 = \vec{x}^2$ with the help of Eqs. (14) and (18)–(23). The 16 parameters appearing in the transformation S_{16} can therefore be called the Cayley–Klein parameters of the de Sitter groups $SO(4, 1)$ and $SO(3, 2)$.

We observe that the existence of a four-dimensional irrep of the five-dimensional rotations group was already pointed out by Pauli (1933) in connection with his investigations on the unification of gravity and electromagnetism in a five-dimensional projective space of real coordinates.

It can be demonstrated that there exists a two-to-one homomorphism between the group S_{16} and the groups $SO(4, 1)$ and $SO(3, 2)$, with the sets of parameters in Eqs. (25) and (26), respectively. The respective proof is exactly analogous to that of Wigner (1959), concerning the groups $SU(2, C)$ and $SO(3)$.

We are dealing with four-dimensional, double-valued irreps S_{16} of the groups $SO(4, 1)$ and $SO(3, 2)$, expressed in terms of the respective Cayley–Klein parameters. In the Minkowski subspace the matrix S_{16} reduces to the block-diagonal form with two $SL(2, C)$ matrices along the diagonal. For three-dimensional rotations, i.e. with $a = b = c = d = u = v = w = z = U = V = W = Z = 0$, the matrix S_{16} reduces to the block-diagonal form with two $SU(2, C)$ matrices along the diagonal.

3. QUASI-ROTATIONS IN FIVE DIMENSIONS

From Eqs. (14)–(16) we determine the following expressions for the change of components of a five-vector under rotations in $E_{4,1}$ and $E_{3,2}$ spaces:

$$\begin{aligned} x_1' = (&A^2 - B^2 - C^2 + D^2 - a^2 + b^2 + c^2 - d^2 \\ &+ U^2 - V^2 - W^2 + Z^2 - u^2 + v^2 + w^2 - z^2)x_1 \\ &+ 2(AB - CD - ab + cd + UV + WZ - uv - wz)x_2 \end{aligned}$$

$$+ 2(AC + BD - ac - bd + UW - VZ - uw + vz)x_3$$
$$+ 2(Ad - Da - Bc + Cb + Uz + Vw - Wv - Zu)x_4$$
$$+ 2(Cv - Az - Bw - Du + aZ + bW - cV + dU)x_5 \qquad (27)$$

$$x_2' = 2(AB + CD - ab - cd + UV - WZ - uv + wz)x_1$$
$$+ (D^2 - A^2 + B^2 - C^2 + a^2 - b^2 + c^2 - d^2$$
$$+ Z^2 - U^2 + V^2 - W^2 + u^2 - v^2 + w^2 - z^2)x_2$$
$$+ 2(BC - AD + ad - bc + UZ + VW - uz - vw)x_3$$
$$+ 2(Ac + Bd - Ca - Db - Uw + Vz + Wu + Zv)x_4$$
$$+ 2(Aw - Bz - Cu - Dv - aW + bZ + cU + dV)x_5 \qquad (28)$$

$$x_3' = 2(AC - BD - ac + bd + UW + VZ - uw - vz)x_1$$
$$+ 2(BC + AD - ad - bc - UZ + VW + uz - vw)x_2$$
$$+ (D^2 - A^2 - B^2 + C^2 + a^2 + b^2 - c^2 - d^2$$
$$+ Z^2 - U^2 - V^2 + W^2 + u^2 + v^2 - w^2 - z^2)x_3$$
$$+ 2(Ba - Ab + Dc - Cd - Uv + Vu + Wz - Zw)x_4$$
$$+ 2(Bu - Av + Cz + Dw - aV + bU + cZ + dW)x_5 \qquad (29)$$

$$x_4' = 2(Da - Ad - Bc + Cb + Uz - Vw + Wv - Zu)x_1$$
$$+ 2(Ac - Bd - Ca + Db + Uw + Vz - Wu - Zv)x_2$$
$$+ 2(Ba - Ab - Cd + Dc - Uv + Vu + Wz - Zw)x_3$$
$$+ (A^2 + B^2 + C^2 + D^2 - a^2 - b^2 - c^2 - d^2$$
$$+ u^2 + v^2 + w^2 + z^2 - U^2 - V^2 - W^2 - Z^2)x_4$$
$$+ 2(AU + BV + CW - DZ - au - bv - cw + dz)x_5 \qquad (30)$$

$$x_5' = 2(Cv + Az - Bw + Du + aZ - bW + cV + dU)x_1$$
$$+ 2(Aw + Bz - Cu + Dv + aW + bZ - cU + dV)x_2$$
$$+ 2(Bu - Av + Cz + Dw - aV + bU + cZ + dW)x_3$$
$$+ 2(DZ - AU - BV - CW - au - bv - cw + dz)x_4$$

$$+ (A^2 + B^2 + C^2 + D^2 + a^2 + b^2 + c^2 + d^2$$
$$- U^2 - V^2 - W^2 - Z^2 - u^2 - v^2 - w^2 - z^2)x_5 \quad (31)$$

The parameters A, B, C, D are real, the parameters a, b, c, d are imaginary, and each of the two tetrads U, V, W, Z and u, v, w, z is either real or imaginary, according to Eqs. (25) and (26).

The five-dimensional rotations can be written in the matrix form

$$x'_{\mathrm{m}} = \mathcal{A}_{\mathrm{m}}\, x_{\mathrm{m}} \quad (32)$$

The respective cracovian form is

$$x'_{\mathrm{c}} = x_{\mathrm{c}} \cdot T\mathcal{A}_{\mathrm{c}} = x_{\mathrm{c}} \cdot P_{\mathrm{c}} \quad (33)$$

The column matrices x'_{m} and x_{m} in Eq. (32) are identical with the column cracovians x'_{c} and x_{c} in Eq. (33), where T denotes the “transpose” cracovian, and the square matrix $\mathcal{A}_{\mathrm{m}}$ in Eq. (32) is identical with the square cracovian $\mathcal{A}_{\mathrm{c}}$ in Eq. (33). The lower indices m and c distinguish square or column tables undergoing the matrix product from those undergoing the cracovian product. The cracovian P_{c} may, for brevity, be called a quasi-rotation cracovian, since it represents the nonassociative transformations connected with the five-dimensional rotations. The elements of the cracovian P_{c} are defined in Eqs. (32) and (33).

4. DOUBLE-VALUED CRACOVIAN REPRESENTATIONS OF THE QUASI-ROTATIONS

We will determine double-valued, eight-dimensional cracovian irreps of the quasi-rotations P_{c} connected with the $SO(3, 2)$ and $SO(4, 1)$ groups. The method of calculation is analogous to that applied in the determination of a double-valued cracovian irrep of the quasi-rotations connected with the proper orhochronous Lorentz group (Kociński, 2001). We firstly consider the nonassociative sedenion group. The γ-symbols in the dot product on the left-hand side of Eq. (5) will be written with a caret, which means that $(\gamma_{\mu\nu\cdots\sigma}) \cdot (\gamma_{\epsilon\eta\cdots\rho})$ will be replaced by $(\hat{\gamma}_{\mu\nu\cdots\sigma}) \cdot (\hat{\gamma}_{\epsilon\eta\cdots\rho})$. The respective nonassociative Clifford algebra has the basis consisting of the right identity τ, the generators $\hat{\gamma}_\mu$, $\mu = 1, \ldots, 4$, fulfilling the condition $\hat{\gamma}_\mu \cdot \hat{\gamma}_\nu + \hat{\gamma}_\nu \cdot \hat{\gamma}_\mu = -2\tau\delta_{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$, and all linearly independent products of these generators. The dimension of this algebra is 2^N (Kociński, 2001). The Cayley table of the nonassociative sedenion group is given in Table I.

The nonassociative sedenion group has a four-dimensional cracovian irrep which is identical with the respective matrix irrep of the sedenion group. It also

Table I. Multiplication Table of the Nonassociative Sedenion Group[a]

	τ	23	31	12	14	24	34	5	15	25	35	45	1	2	3	4
τ	τ	32	13	21	41	42	43	5	15	25	35	45	−1	−2	−3	−4
23	23	τ	21	31	−5	43	24	41	−4	35	−25	1	45	−3	2	−15
31	31	12	τ	32	34	−5	41	42	−35	−4	15	2	3	45	−1	−25
12	12	13	23	τ	42	14	−5	43	25	−15	−4	3	−2	1	45	−35
14	14	−5	43	24	τ	21	31	32	45	−3	2	−15	−4	35	−25	1
24	24	34	−5	41	12	τ	32	13	3	45	−1	−25	−35	−4	15	2
34	34	42	14	−5	13	23	τ	21	−2	1	45	−35	25	−15	−4	3
5	5	14	24	34	23	31	12	τ	1	2	3	4	−15	−25	−35	−45
15	15	4	−35	25	45	−3	2	−1	$-\tau$	12	13	14	−5	43	24	32
25	25	35	4	−15	3	45	−1	−2	21	$-\tau$	23	24	34	−5	41	13
35	35	−25	15	4	−2	1	45	−3	31	32	$-\tau$	34	42	14	−5	21
45	45	−1	−2	−3	−15	−25	−35	−4	41	42	43	$-\tau$	23	31	12	−5
1	1	45	−3	2	4	−35	25	−15	−5	43	24	32	$-\tau$	12	13	14
2	2	3	45	−1	35	4	−15	−25	34	−5	41	13	21	$-\tau$	23	24
3	3	−2	1	45	−25	15	4	−35	42	14	−5	21	31	32	$-\tau$	34
4	4	−15	−25	−35	−1	−2	−3	−45	23	31	12	−5	41	42	43	$-\tau$

[a] Here τ denotes the right unit element and numbers denote the indices of the respective $\hat{\gamma}$-symbols. We have $\hat{\gamma}_{\mu\nu} = -\hat{\gamma}_{\nu\mu},\ \mu \neq \nu,\ \mu, \nu = 1, 2, 3, 4$; and $\hat{\gamma}_{\mu 5} = \hat{\gamma}_{5\mu},\ \mu = 1, 2, 3, 4$.

has, however, an eight-dimensional cracovian irrep of the form

$$\hat{\gamma}_{23} = \left\{\begin{array}{cccc|cccc} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0 \end{array}\right\} \quad \hat{\gamma}_{31} = \left\{\begin{array}{cccc|cccc} 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{array}\right\} \tag{34}$$

$$\hat{\gamma}_{12} = \left\{\begin{array}{cccc|cccc} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0 \end{array}\right\} \quad \hat{\gamma}_{14} = \left\{\begin{array}{cccc|cccc} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{array}\right\} \tag{35}$$

$$\hat{\gamma}_{24} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{array} \right\} \quad \hat{\gamma}_{34} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right\} \tag{36}$$

$$\hat{\gamma}_{5} = \left\{ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array} \right\} \quad \hat{\gamma}_{15} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\} \tag{37}$$

$$\hat{\gamma}_{25} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ \hline i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \end{array} \right\} \quad \hat{\gamma}_{35} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ \hline 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \end{array} \right\} \tag{38}$$

$$\hat{\gamma}_{45} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ \hline 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\} \quad \hat{\gamma}_{1} = \left\{ \begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\} \tag{39}$$

$$\hat{\gamma}_2 = \left\{ \begin{array}{cccc|cccc} 0&0&0&0&i&0&0&0\\ 0&0&0&0&0&-i&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&-i\\ \hline -i&0&0&0&0&0&0&0\\ 0&i&0&0&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&0&0&i&0&0&0&0 \end{array} \right\} \quad \hat{\gamma}_3 = \left\{ \begin{array}{cccc|cccc} 0&0&0&0&0&i&0&0\\ 0&0&0&0&i&0&0&0\\ 0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&-i&0\\ \hline 0&-i&0&0&0&0&0&0\\ -i&0&0&0&0&0&0&0\\ 0&0&0&i&0&0&0&0\\ 0&0&i&0&0&0&0&0 \end{array} \right\} \tag{40}$$

$$\hat{\gamma}_4 = \left\{ \begin{array}{cccc|cccc} 0&0&0&0&0&0&-i&0\\ 0&0&0&0&0&0&0&i\\ 0&0&0&0&i&0&0&0\\ 0&0&0&0&0&-i&0&0\\ \hline 0&0&-i&0&0&0&0&0\\ 0&0&0&i&0&0&0&0\\ i&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0 \end{array} \right\} \tag{41}$$

Here and in the following we use wavy brackets for cracovian tables to distinguish them from matrix tables.

It suffices to prove the irreducibility of this cracovian representation into two four-dimensional cracovian representations. If it is reducible then only to this form, since the four-dimensional cracovian representation of the nonassociative biquaternion subgroup of the nonassociative sedenion group has already been proved to be irreducible (Kociński, 2001).

Firstly, we observe that the condition that any similarity transformation S of a cracovian representation has to preserve the Cayley table of the respective nonassociative group G', namely

$$(S \cdot TA \cdot S^{-1}) \cdot (S \cdot TB \cdot S^{-1}) = S \cdot T(A \cdot B) \cdot S^{-1} \tag{42}$$

for any A and B belonging to the nonassociative group G', leads to the two conditions

$$S \cdot B \cdot S^{-1} = S^{-1} \cdot B \cdot S \quad \text{and} \quad S \cdot TB \cdot S^{-1} = S^{-1} \cdot TB \cdot S \tag{43}$$

for any cracovian B belonging to the nonassociative group G', from which follows that

$$S^{-1} = S \tag{44}$$

for any cracovian similarity transformation S which preserves the Cayley table.

Secondly, we observe that from the definition of the nonassociative product in Eq. (5) it follows that every matrix representation of a group at the same time is a cracovian representation of the respective nonassociative group and vice versa. A matrix irrep turns into a cracovian irrep; however, a cracovian irrep can turn into a reducible matrix representation. An example of such a situation was given (Kociński, 2001). If the eight-dimensional cracovian representation Q_{16} were reducible to two four-dimensional representations, the respective two diagonal blocks would at the same time belong to a matrix representation of the sedenion group. The sedenion group has only one four-dimensional matrix irrep. A further transformation of the block-diagonal form of the cracovians Q_{16} then would turn the blocks into a particular standard form of the four-dimensional matrix representation. With S denoting the resultant similarity transformation we therefore can write

$$S \cdot TC(\hat{\gamma}_\nu) \cdot S = C'(\hat{\gamma}_\nu) \tag{45}$$

where $C(\hat{\gamma}_\nu)$ denotes the original eight-dimensional cracovian and $C'(\hat{\gamma}_\nu)$ the block-diagonal cracovian, whose four-dimensional table is identical with the matrix table representing γ_ν in the chosen matrix representation of the sedenion group. Assume that the four-dimensional matrix irrep of the sedenion group is that in Flügge (1964) or Gross (1993). It then can be shown that a similarity transformation S fulfilling Eq. (45) does not exist. This means that the eight-dimensional cracovian representation Q_{16} is irreducible.

We can now determine a double-valued, eight-dimensional cracovian irrep of each of the nonassociative groups of quasi-rotations P_{c} in Eq. (33), in $(3 + 2)$- and $(4 + 1)$-dimensional pseudo-orthogonal spaces.

In $(3 + 2)$ dimensions we introduce the nonassociative sedenion transformation

$$\begin{aligned} Q(3, 2) = {} & A\hat{\gamma}_{23} + B\hat{\gamma}_{31} + C\hat{\gamma}_{12} + D\tau + a\hat{\gamma}_{14} + b\hat{\gamma}_{24} + c\hat{\gamma}_{34} + d\hat{\gamma}_5 \\ & - iu\hat{\gamma}_{15} - iv\hat{\gamma}_{25} - iw\hat{\gamma}_{35} + Z\hat{\gamma}_{45} + U\hat{\gamma}_1 + V\hat{\gamma}_2 + W\hat{\gamma}_3 - iz\hat{\gamma}_4 \end{aligned} \tag{46}$$

where, for convenience, the parameters u, v, w, z as well as the parameters U, V, W, Z now are real. This means that we have replaced the imaginary parameters u, v, w, z in Eq. (26) by iu, iv, iw, iz, respectively, with real u, v, w, z. The parameters A, B, C, D are real and the parameters a, b, c, d are imaginary according to their primary definition in Eq. (10).

In $(4 + 1)$ dimensions, shifting i in Eq. (46) from u, v, w, z to U, V, W, Z, and at the same time reversing the signs of u, v, w, z, U, V, W, Z, we obtain from Eq. (46) the following transformation:

$$\begin{aligned} Q(4, 1) = {} & A\hat{\gamma}_{23} + B\hat{\gamma}_{31} + C\hat{\gamma}_{12} + D\tau + a\hat{\gamma}_{14} + b\hat{\gamma}_{24} + c\hat{\gamma}_{34} + d\hat{\gamma}_5 \\ & + u\hat{\gamma}_{15} + v\hat{\gamma}_{25} + w\hat{\gamma}_{35} - iZ\hat{\gamma}_{45} - iU\hat{\gamma}_1 - iV\hat{\gamma}_2 - iW\hat{\gamma}_3 + z\hat{\gamma}_4 \end{aligned} \tag{47}$$

Introducing for the $\hat{\gamma}_{\nu\mu}$ and $\hat{\gamma}_{\nu}$ the cracovian irrep Q_{16} in Eqs. (34)–(41) we obtain the respective cracovian forms of these nonassociative transformations.

In (3 + 2) dimensions we have

$$Q(3,2) = \left\{\begin{array}{cccccccc} (D+d) & (A+a) & -(B+b) & (C+c) & (iV+v) & (iW+w) & -(iZ+z) & -(iU+u) \\ -(A+a) & (D+d) & -(C+c) & -(B+b) & (iW+w) & -(iV+v) & -(iU+u) & (iZ+z) \\ (B+b) & (C+c) & (D+d) & -(A+a) & (iZ+z) & -(iU+u) & (iV+v) & -(iW+w) \\ -(C+c) & (B+b) & (A+a) & (D+d) & -(iU+u) & -(iZ+z) & -(iW+w) & -(iV+v) \\ -(iV-v) & -(iW-w) & (iZ-z) & (iU-u) & (D-d) & (A-a) & -(B-b) & (C-c) \\ -(iW-w) & (iV-v) & (iU-u) & -(iZ-z) & -(A-a) & (D-d) & -(C-c) & -(B-b) \\ -(iZ-z) & (iU-u) & -(iV-v) & (iW-w) & (B-b) & (C-c) & (D-d) & -(A-a) \\ (iU-u) & (iZ-z) & (iW-w) & (iV-v) & -(C-c) & (B-b) & (A-a) & (D-d) \end{array}\right\} \tag{48}$$

with the inverse cracovian $Q^{-1}(3,2)$ in the form

$$Q^{-1}(3,2) = \left\{\begin{array}{cccccccc} (D+d) & (A+a) & -(B+b) & (C+c) & -(iV+v) & -(iW+w) & (iZ+z) & (iU+u) \\ -(A+a) & (D+d) & -(C+c) & -(B+b) & -(iW+w) & (iV+v) & (iU+u) & -(iZ+z) \\ (B+b) & (C+c) & (D+d) & -(A+a) & -(iZ+z) & (iU+u) & -(iV+v) & (iW+w) \\ -(C+c) & (B+b) & (A+a) & (D+d) & (iU+u) & (iZ+z) & (iW+w) & (iV+v) \\ (iV-v) & (iW-w) & -(iZ-z) & -(iU-u) & (D-d) & (A-a) & -(B-b) & (C-c) \\ (iW-w) & -(iV-v) & -(iU-u) & (iZ-z) & -(A-a) & (D-d) & -(C-c) & -(B-b) \\ (iZ-z) & -(iU-u) & (iV-v) & -(iW-w) & (B-b) & (C-c) & (D-d) & -(A-a) \\ -(iU-u) & -(iZ-z) & -(iW-w) & -(iV-v) & -(C-c) & (B-b) & (A-a) & (D-d) \end{array}\right\} \tag{49}$$

In (4 + 1) dimensions, shifting i in Eq. (48) from U, V, V, Z to u, v, w, z, respectively, we obtain the cracovian

$$Q(4,1) = \left\{\begin{array}{cccccccc} (D+d) & (A+a) & -(B+b) & (C+c) & (V+iv) & (W+iw) & -(Z+iz) & -(U+iu) \\ -(A+a) & (D+d) & -(C+c) & -(B+b) & (W+iw) & -(V+iv) & -(U+iu) & (Z+iz) \\ (B+b) & (C+c) & (D+d) & -(A+a) & (Z+iz) & -(U+iu) & (V+iv) & -(W+iw) \\ -(C+c) & (B+b) & (A+a) & (D+d) & -(U+iu) & -(Z+iz) & -(W+iw) & -(V+iv) \\ -(V-iv) & -(W-iw) & (Z-iz) & (U-iu) & (D-d) & (A-a) & -(B-b) & (C-c) \\ -(W-iw) & (V-iv) & (U-iu) & -(Z-iz) & -(A-a) & (D-d) & -(C-c) & -(B-b) \\ -(Z-iz) & (U-iu) & -(V-iv) & (W-iw) & (B-b) & (C-c) & (D-d) & -(A-a) \\ (U-iu) & (Z-iz) & (W-iw) & (V-iv) & -(C-c) & (B-b) & (A-a) & (D-d) \end{array}\right\} \tag{50}$$

where u, v, w, z, U, V, W, Z are real. Shifting i from U, V, W, Z to u, v, w, z, respectively, in the cracovian $Q^{-1}(3, 2)$ in Eq. (49), we obtain the inverse cracovian $Q^{-1}(4, 1)$. We observe that the cracovian $Q^{-1}(3, 2)$ is the inverse of the cracovian $Q(3, 2)$ when the conditions in Eqs. (18)–(23) are applied. The same holds for the cracovians $Q^{-1}(4, 1)$ and $Q(4, 1)$.

We will now show that the cracovians $Q(3, 2)$ and $Q(4, 1)$ yield double-valued representations of the nonassociative groups of transformations P_c defined in Eq. (33), connected with the respective groups of rotations $SO(3, 2)$ and $SO(4, 1)$.

We begin with rewriting the column cracovian x_c in Eq. (33) in a square cracovian form, to be denoted by X_c. To this end we write

$$X_c = x_1\hat{\gamma}_{23} + x_2\hat{\gamma}_{31} + x_3\hat{\gamma}_{12} + x_4\hat{\gamma}_5 + x_5\hat{\gamma}_4 \tag{51}$$

and replace the $\hat{\gamma}$'s by the respective cracovians in Eqs. (34)–(41), thus obtaining the eight-dimensional cracovian

$$X_c = \left\{\begin{array}{rrrr|rrrr} x_4 & x_1 & -x_2 & x_3 & 0 & 0 & ix_5 & 0 \\ -x_1 & x_4 & -x_3 & -x_2 & 0 & 0 & 0 & -ix_5 \\ x_2 & x_3 & x_4 & -x_1 & -ix_5 & 0 & 0 & 0 \\ -x_3 & x_2 & x_1 & x_4 & 0 & ix_5 & 0 & 0 \\ \hline 0 & 0 & ix_5 & 0 & -x_4 & x_1 & -x_2 & x_3 \\ 0 & 0 & 0 & -ix_5 & -x_1 & -x_4 & -x_3 & -x_2 \\ -ix_5 & 0 & 0 & 0 & x_2 & x_3 & -x_4 & -x_1 \\ 0 & ix_5 & 0 & 0 & -x_3 & x_2 & x_1 & -x_4 \end{array}\right\} = (\vec{x}_c, \vec{q}_c) \tag{52}$$

where $(\vec{x}_c, \vec{q}_c)$ denotes a scalar product of $\vec{x}_c$ and $\vec{q}_c = (\hat{\gamma}_{23}, \hat{\gamma}_{31}, \hat{\gamma}_{12}, \hat{\gamma}_5, \hat{\gamma}_4)$.

We now consider the nonassociative transformation

$$Q \cdot (T X_c) \cdot Q^* = X'_c = (\vec{x}'_c, \vec{q}_c) \tag{53}$$

where Q stands for the cracovians $Q(3, 2)$ or $Q(4, 1)$ in Eqs. (48) and (50), respectively, and where * denotes the conjugate complex operation. It can be verified that the coordinates $x'_1, x'_2, x'_3, x'_4, x'_5$ calculated from Eq. (53) are those calculated from Eq. (33). The elements of the cracovian P_c are equal to the elements of the transposed table $\mathcal{A}_c$ which is identical with the matrix table $\mathcal{A}_m$ in Eq. (32). Consequently, the first index of the element a_{jk} of the matrix $\mathcal{A}_m$ is the column index of the cracovian table P_c. This means that the transformation in Eq. (33) which carries x_c into $x'_c = x_c \cdot P_c$ can also be determined from Eq. (53).

We observe that in establishing this we made use of the following property of the six conditions for the Cayley–Klein parameters in Eqs. (18)–(23). These equations serve for the determination of the 6 nonrotational parameters D, d, U, V, W, z in terms of the 10 rotational parameters $A, B, C, a, b, c, u, v, w, Z$. It can be shown that the expressions for the nonrotational parameters are polynomes of the second degree in the rotational parameters. Consequently, these expressions are independent of the simultaneous change of sign of all the rotational parameters. This means that the six conditions can be applied with the same result, independently of the reversal of sign of all the rotational parameters appearing in them.

It can be demonstrated that to the product $Q_1 \cdot Q_2$ of two transformations of the type $Q(3, 2)$ or $Q(4, 1)$ corresponds the product of the respective quasi-rotations $P_c(Q_1) \cdot P_c(Q_2) = P_c(Q_1 \cdot Q_2)$. The proof is analogous to that in the case of quasi-rotations in the Minkowski space (Kociński, 2001). There exists a two-to-one homomorphism between the nonassociative groups of eight-dimensional cracovians in Eqs. (48) and (50) and the respective nonassociative groups of quasi-rotations P_c in Eq. (33).

5. THE COVARIANCE OF THE DIRAC EQUATION

Dirac's electron-wave equation (Dirac, 1935) in de Sitter space was investigated in a number of papers. We only name the work of Gürsey and Lee (1963) and the recent survey article of Halpern (2001). Dirac's paper contains also an alternative form of his electron-wave equation in Minkowski space. This form served as a basis for constructing a five-dimensional electron-wave equation in (4 + 1)- or (3 + 2)-dimensional, pseudo-orthogonal space (Kociński, 1999, 2000). The five-dimensional equation is given by

$$[\gamma_\mu(\partial_\mu - i a_\mu) - i\gamma_5\kappa]u = 0 \tag{54}$$

with $a_\mu = (e/\hbar c)A_\mu$, $\mu = 1, 2, 3, 4$, and $a_5 = m\chi/\hbar c$ or $a_5 = im\chi/\hbar c$ for x_5 real or x_5 imaginary, respectively, with χ denoting a real nonelectromagnetic scalar potential, while $\kappa = mc/\hbar$. When Eq. (54) is delimited to the Minkowski space ($\mu = 1, 2, 3, 4$), it represents the alternative form of the Dirac equation (Dirac, 1935).

A solution u of Eq. (54) can be referred to the basis of the sedenion algebra without applying a matrix representation for the algebra generators (Sommerfeld, 1944). It then has the form

$$u = c_0(\vec{x}) + \gamma_1 c_1(\vec{x}) + \cdots + \gamma_1\gamma_2\gamma_3\gamma_4 c_{15}(\vec{x}) \tag{55}$$

where $c_0(\vec{x}), \ldots, c_{15}(\vec{x})$ are complex functions of a four-vector $\vec{x} = (x_1, x_2, x_3, x_4)$ in the Minkowski subspace, or of a five-vector $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ in the two considered five-dimensional spaces.

The covariance of Eq. (54) with respect to rotations belonging to the $SO(4, 1)$ and $SO(3, 2)$ groups was demonstrated in Kociński (1999, 2000). To this end Eq. (54) was rewritten in the form

$$\gamma_\mu \mathcal{D}_\mu u = 0 \tag{56}$$

with $\mathcal{D}_\mu = \partial_\mu - i\Omega_\mu$ or $\mathcal{D}_\mu = p_\mu - i a_\mu$ when x_5 is real or imaginary, respectively, where $\Omega_\mu = a_\mu$, $\mu = 1, \ldots, 4$, and $\Omega_5 = a_5 + \kappa$ for x_5 real, and where $p_\mu = \partial_\mu$, $\mu = 1, \ldots, 4$, and $p_5 = \partial_5 - i\kappa$ for x_5 imaginary.

We will now show that the five-dimensional form of the Dirac equation in Eq. (54) or in Eq. (56) is covariant with respect to the quasi-rotations of the quasigroups connected with the groups $SO(4, 1)$ and $SO(3, 2)$. To this end we rewrite Eq. (56) in the basis of the nonassociative Clifford algebra (Kociński, 2001). It then acquires the form

$$\hat{u} \cdot [(\tau \hat{\gamma}_\mu)\mathcal{D}_\mu] = 0 \tag{57}$$

where $\hat{u}$ is obtained from u in Eq. (55) by replacing γ_μ with $\hat{\gamma}_\mu$ and $c_0(\vec{x})$ by $\tau c_0(\vec{x})$, and where to avoid unnecessary brackets the dot between τ and the symbol to the right of it has been omitted (Kociński, 2001). This means that in Eq. (57), $(\tau \cdot \hat{\gamma}_\mu)$ has been replaced by $(\tau \hat{\gamma}_\mu)$, and $\tau \cdot \hat{\gamma}_5$ has been replaced by $\tau \hat{\gamma}_5$. In the rotated coordinate system we are dealing with the "primed" quantities: (i) $\hat{u}'(\vec{x}') = \hat{u}(\vec{x}) \cdot \tau \hat{V}$, where again $\tau \hat{V}$ stands for $(\tau \cdot \hat{V})$, and (ii) $\mathcal{D}'_\mu$. The transformation $\hat{V}$ depends on the 16 basis elements of the nonassociative Clifford algebra and on the parameters of the quasi-rotation P_c in Eq. (33). In the rotated coordinate system we thus obtain the equation

$$(\hat{u} \cdot \tau \hat{V}) \cdot [(\tau \hat{\gamma}_\nu)\mathcal{D}'_\nu] = 0 \tag{58}$$

where again $(\hat{u} \cdot \tau \hat{V})$ replaces $[\hat{u} \cdot (\tau \cdot \hat{V})]$. Multiplying this equation from the right by $\hat{V}^{-1}$ we obtain

$$(\hat{u} \cdot \tau \hat{V}) \cdot [(\tau \hat{\gamma}_\nu)\mathcal{D}'_\nu] \cdot \hat{V}^{-1} = 0 \tag{59}$$

By a repeated application of Eq. (6) and of the equality $\tau(\hat{a} \cdot \hat{b}) = \hat{b} \cdot \hat{a}$, which is valid for any two elements $\hat{a}$ and $\hat{b}$ of the nonassociative group algebra (Kociński, 2001), this equation is transformed to the form

$$\hat{u} \cdot \{\hat{V}^{-1} \cdot [\hat{\gamma}_\nu \mathcal{D}'_\nu] \cdot \hat{V}\} = 0 \tag{60}$$

We now return to Eq. (57) and express in it $\mathcal{D}_\mu$ through $\mathcal{D}'_\nu$, utilizing the cracovian formula

$$\mathcal{D}_\mu = \mathcal{D}'_\nu \left(\mathcal{A}_\mathrm{c}^{-1}\right)_{\mu\nu} \tag{61}$$

with the transformation $\mathcal{A}_\mathrm{c}$ defined in Eq. (33), thus obtaining the equation

$$\hat{u} \cdot \left\{(\tau \hat{\gamma}_\mu)\mathcal{D}'_\nu \left(\mathcal{A}_\mathrm{c}^{-1}\right)_{\mu\nu}\right\} = 0 \tag{62}$$

The comparison of Eqs. (60) and (62) yields the covariance condition

$$(\hat{V}^{-1} \cdot \hat{\gamma}_\nu) \cdot \hat{V} = \sum_\mu (\tau \hat{\gamma}_\mu)\left(\mathcal{A}_\mathrm{c}^{-1}\right)_{\mu\nu} \tag{63}$$

Utilizing on the left-hand side the relation $ABC = C \cdot TB \cdot TA$ between the matrix product and the cracovian product of the three tables A, B, and C (Kociński, 2001), and on the right-hand side the equality $\mathcal{A}_{\mathrm{c}}^{-1} = \mathcal{A}_{\mathrm{m}}$, which is valid for the pseudo-orthogonal rotations $\mathcal{A}_{\mathrm{m}}$, we can verify that Eq. (63) transforms to the customary matrix covariance condition.

6. A HIDDEN SYMMETRY OF THE DIRAC EQUATION

Weyl (1952) introduced the notion of a "hidden" symmetry group of an object. He considered a set with a symmetry group G. This can be the set of all roots of a polynom, the set of space-time points or of all nods of a crystal lattice. He showed that essential features of a set endowed with structure can be determined by studying the group of all automorphisms *Aut* G of this set which preserve all its structural relations. The group G determines the "obvious" symmetry and the group *Aut* G, the "hidden" symmetry of the set. The concept of a hidden symmetry group was discussed from the standpoint of group actions on sets by Florek *et al.* (1988) and Lulek *et al.* (1995).

In one of the examples discussed by Weyl (1952), the object is a regular septadecagon. The property under consideration is the possibility of construction of that regular septadecagon with the help of a compass and a ruler. The symmetry group G of the set of vertices of a regular septadecagon is the cyclic group C_{17}. This is the obvious geometric symmetry group of the regular septadecagon. The vertices of a regular septadecagon are determined by the roots of the equation $z^{17} - 1 = 0$, with $z = x + iy$. One of the roots is $z = 1$, and the remaining 16 roots are determined by an algebraic equation of degree 16. The root $z = 1$ determines the starting vertex in the construction of a regular septadecagon. The determination of the positions of the remaining 16 vertices is shown to be connected with the group of permutations of the 16 roots of the algebraic equation of the degree 16. This appears to be the cyclic group C_{16}; hence *Aut* $C_{17} = C_{16}$. This is the hidden symmetry group of a regular septadecagon. The possibility of construction of a regular septadecagon with the help of a compass and a ruler hinges on the group C_{16}.

In our case the object is the Dirac equation. The de Sitter groups $SO(4, 1)$ and $SO(3, 2)$, or their subgroup, i.e. the proper orthochronous Lorentz group, are the *obvious* geometric symmetry groups of that equation in a five-dimensional space or in Minkowski space, respectively. This equation is referred to the generators of the sedenion algebra and its solutions (cf. Eq. (55)) to the basis of that algebra. Weyl's set endowed with a structure now is constituted by the basis elements of the sedenion algebra. The structural condition for the algebra generators is given in Eq. (1). This structural condition is preserved by the two groups of automorphisms in Eqs. (2) and (3). A solution u in Eq. (55) of the Dirac equation is covariant with respect to the two groups of automorphisms in Eqs. (2) and (3). In turn the Dirac equation is invariant under these automorphisms, since the four (or five, when γ_5 is introduced), mutually anticommuting γ_μ's which appear in it obey the defining condition in Eq. (1) in their original forms as well as after the transformations in Eqs. (2) and (3). We notice that the two groups of automorphisms in Eqs. (2) and (3) correspond to the group automorphism $Aut\, C_{17} = C_{16}$ in the case of a regular septadecagon. These, however, do not determine by themselves a hidden symmetry of the Dirac equation. Further steps are required for disclosing a possible hidden symmetry. Firstly, on the basis of those groups of automorphisms, we have to define the nonassociative product in Eq. (5) and next to determine the nonassociative group of quasi-rotations P_c in Eq. (33). We have shown that the Dirac equation is covariant with respect to those quasi-rotations, and it seems that the nonassociative group of quasi-rotations P_c may, perhaps, be recognized as a hidden symmetry of the Dirac equation.

Let us suppose that the analogy holds between Weyl's hidden symmetry group and the symmetry of the Dirac equation with respect to the nonassociative group P_c. We could then expect that some as yet undisclosed properties of the Dirac equation might follow from that hidden symmetry. It seems that the double-valued cracovian irreps $Q(4, 1)$ or $Q(3, 2)$ of the respective nonassociative groups of quasi-rotations could be examined in this respect.

7. CONCLUSIONS

Continuing the investigations of Sommerfeld (1944) and of Gürsey (1964) we have determined the Cayley–Klein parameters for the de Sitter groups and also the four-dimensional, double-valued irreps of those groups. These irreps represent an analogue of the $SL(2, C)$ double-valued irrep of the proper orthochronous Lorentz group. These results serve as a basis of the second part of this paper, which is an extension of the investigations of quasigroups connected with the quasi-rotations in the Minkowski space (Kociński, 2001). We have determined the quasi-rotations nonassociative groups connected with the de Sitter groups $SO(4, 1)$ and $SO(3, 2)$. A particular type of a quasigroup connected with the sedenion group was determined. This served as a basis for determining double-valued, eight-dimensional cracovian irreps for those quasigroups. In the Minkowski subspace of the $(3 + 2)$- or $(4 + 1)$-dimensional, pseudo-orthogonal spaces, the double-valued cracovian irreps reduce to the double-valued cracovian irrep for the quasi-rotations connected with the proper orthochronous Lorentz group (Kociński, 2001). A possible application in physics of the quasigroups investigated here may, perhaps, be sought with the reference to the Dirac equation or to its five-dimensional form (Kociński, 1999, 2000). The covariance of the Dirac equation with respect to the nonassociative group of quasi-rotations connected with proper orthochronous Lorentz group, or of its five-dimensional form with respect the the nonassociative groups of quasi-rotations connected with the de Sitter groups was demonstrated. An analogy has been drawn between a quasigroup connected with a group and Weyl's hidden symmetry group connected with the obvious symmetry group of an object. If the object is the Dirac equation with its obvious symmetry determined by the proper orthochronous Lorentz group, or the $SO(4,1)$ and $SO(3,2)$ groups in the case of its five-dimensional extension, the respective nonassociative groups of quasi-rotations may be the analogues of hidden symmetry groups. They may determine a hidden symmetry of the Dirac equation. The property or properties under consideration which depend on this hidden symmetry of the Dirac equation could be, perhaps, some hitherto undisclosed, latent dynamical consequences, connected with the double-valued cracovian irreps of the nonassociative groups of quasi-rotations.

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